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NAVIER–STOKES EQUATIONS WITH IMPOSED PRESSURE AND VELOCITY FLUXES

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SUMMARY

Boundary value problems for Stokes and Navier–Stokes equations with non-standard boundary conditions are studied. Included is the case where the pressure or its normal derivative is given on some part of the boundary or the pressure is given up to a constant but given velocity flux. First, a variational formulation is introduced which is shown to be equivalent to the Stokes equations with the non-standard boundary conditions under consideration. The existence and uniqueness of the solution of the variational problem are studied. Secondly, most of the results obtained for the Stokes equations are extended to the case of the Navier–Stokes equations. The final section is devoted to numerical experiments, flows in pipes and physiological flows.

KEY WORDS velocity–pressure formulation; pressure boundary condition; single and branched pipes; steady and periodic flows

1. INTRODUCTION

This paper is concerned with the stationary Stokes and Navier–Stokes equations with non-standard boundary conditions. Specifically, the case where the pressure is given on some part of the boundary will be considered.

To be precise, the flow of a *viscous* incompressible fluid which occupies a bounded domain Ω of $\mathbb{R}^3$ is studied. The velocity $\mathbf{u}$ and pressure p are assumed to satisfy in this domain Ω either the stationary Stokes equations

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{1b}$$

or the stationary Navier–Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{2b}$$

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In both cases ν is positive and denotes the kinematic viscosity of the fluid and $\mathbf{f}$ denotes the density of the external forces. We do not consider the time-dependent problem, but all the results extend to the operator found after a time discretization, i.e. when $-\nu\Delta\mathbf{u}$ is replaced by $(1/\delta t)\mathbf{u} - \nu\Delta\mathbf{u}$.

As regards the boundary conditions, they are assumed to be of three different types.

1. The velocity is given on a portion Γ_1 of the boundary of Ω.
2. The pressure and the tangential component of the velocity are given on a second portion Γ_2 of the boundary.
3. The normal component of the velocity and the tangential component of the vorticity are given on the remainder of Γ_3 of the boundary.

In the case of the Stokes problem (1) these boundary conditions read

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1, \tag{3a}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{and} \quad p = p_0 \quad \text{on } \Gamma_2, \tag{3b}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{and} \quad (\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3, \tag{3c}$$

where $\mathbf{u_0}$, $\mathbf{a}$, p_0, $\mathbf{b}$ and $\mathbf{h}$ are given functions and $\mathbf{n}$ denotes the unit outward normal to the boundary of Ω. The data must be compatible so that there exists at least one solenoidal vector which satisfies them.

In the case of the Navier-Stokes problem (2) the boundary conditions read

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1, \tag{4a}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{and} \quad p + \tfrac{1}{2}|\mathbf{u}|^2 = p_0 \quad \text{on } \Gamma_2, \tag{4b}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{and} \quad (\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3. \tag{4c}$$

Note that in the Navier–Stokes problem the dynamic pressure $p + \frac{1}{2}|\mathbf{u}|^2$ plays in (4b) the role played by the static pressure p in (3b) for the Stokes problem. Condition (4b) is involved in the special variational formulation of the Navier–Stokes equations. Using the usual variational formulation with the method of characteristics, the static pressure condition is used (see Remark in Section 4.2.2). A possible application of the above set of boundary conditions associated with a pipe flow is given in Table I.

Moreover, it will be shown in Section 2.7 that for the Stokes problem the normal derivative of the pressure on Γ_3 can be computed explicitly directly from the data of the problem; for the Navier–Stokes problem the normal derivative of the pressure on Γ_3 depends both on the data and on the velocity itself (see Section 3.3).

In our analysis p_0 may also be known only up to a constant, a different constant on each connected component Γ_{2i} of Γ_2; in such cases the fluxes $\int_{\Gamma_{2i}} \mathbf{u} \cdot \mathbf{n}$ must also be given.

Table I. A family of simultaneously applied boundary conditions in a pipe flow, the domain boundary being partitioned into three parts for (1) a Dirichlet condition, homogeneous or not, (2) a pressure condition and (3) a velocity condition, for a well-posed problem († special variational formulation)

Boundary	Conditions	Duct flow application
Γ_1	$\mathbf{u} = \mathbf{0}$	No-slip
Γ_1'	$\mathbf{u} = \mathbf{u}_0$	Injection velocity
Γ_2	$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n}$ and $p = p_0$	Pressure condition at tube exit, with an unknown velocity
	or $p + \frac{1}{2}\lvert\mathbf{u}\rvert^2 = p_0$ †	distribution (monodimensional flow, $\mathbf{a} = \mathbf{0}$)
Γ_3	$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n}$ and $\Delta \times \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$	Jet (dye) (1D ($\mathbf{h} = 0$) or 3D (bend output))

Such problems can be found in many circumstances such as pipe flows. If a pipe bifurcates into two pipes, then conditions on p and/or conditions on fluxes are inevitable in order to monitor the amount of fluid that flows in each pipe. For want of better conditions, engineers have often replaced the conditions on the pressure by a condition on the normal stress $p - \nu\mathbf{n}\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})\mathbf{n}$. If there are no boundary layers, then the two conditions are close to each other when ν is small. This last condition is not suitable for pipe flows, where a boundary layer develops rapidly.

There exists a considerable literature concerning the Stokes and Navier–Stokes equations with a no-slip boundary condition (see e.g. the books by Lions,[1] Tartar[2] and Temam[3]). Other boundary conditions for the Stokes and Navier–Stokes equations (no stress and slip at the boundary) have been recently studied.[4–8] This paper is a summary, stripped of most technical considerations, of a more mathematical paper.[9] It contains, however, more numerical results than Reference 9.

The present contribution to the study of problems (1), (3) and (2), (4) is threefold. Firstly, variational formulations of both problems are given and these are proved to be equivalent to the boundary value problems considered. Secondly, existence and uniqueness are proved for these variational problems. Existence and uniqueness results are also important, because this is the only way to make sure that one has a complete set of boundary conditions (not too few and not too many). Thirdly, a finite element discretization is given for which a classical error estimate applies. As usual, the discretization is given in a space which approximates the solenoidal vectors. Therefore it can be solved in practice by using the velocity–pressure formulation; the pressure is the Lagrange multiplier of the divergence-free constraints on the test functions of the finite element space.

Finally, some numerical examples are given which prove that the method is feasible.

2. THE STOKES EQUATIONS WITH BOUNDARY CONDITIONS INVOLVING THE PRESSURE

2.1. Formulation of the Stokes problem

The boundary Γ of Ω is assumed to be made of three smooth* subsets, which we denote by $\{\Gamma_1, \Gamma_2, \Gamma_3\}$, each with a finite number of connected components, such that

$$\Gamma_1 \neq \varnothing, \tag{5a}$$

$$\Gamma_i \cap \Gamma_j = \varnothing \quad \forall i, j = 1, 2, 3, \quad i \neq j, \tag{5b}$$

$$\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3. \tag{5c}$$

We will denote by $\Gamma_{21}, \ldots, \Gamma_{2r}$ the connected components of Γ_2.

We shall assume some smoothness for $\mathbf{f}$, p_0 and $\mathbf{h}$† and a compatibility condition for the boundary conditions which says that there exists at least one incompressible velocity field which satisfies them.

There exists a function $\mathbf{U}_0 \in H^1(\Omega)^3$ such that:

$$\nabla\cdot\mathbf{U}_0 = 0 \quad \text{in } \Omega \tag{6a}$$

$$\mathbf{U}_0 = \mathbf{u}_0 \quad \text{on } \Gamma_1, \tag{6b}$$

$$\mathbf{U}_0 \times \mathbf{n} = \mathbf{a}\times\mathbf{n} \quad \text{on } \Gamma_2, \tag{6c}$$

$$\mathbf{U}_0 \cdot \mathbf{n} = \mathbf{b}\cdot\mathbf{n} \quad \text{on } \Gamma_3. \tag{6d}$$

* Γ locally Lipschitz and Γ_i open subsets of class $\mathscr{C}^{1,1}$.

† $f \in L^2(\Omega)^3$, $\nabla\cdot f \in L^2(\Omega)$, $\nabla\times f \in L^2(\Omega)^3$, $p_0 \in H^{-1/2}(\Gamma_2)/\mathbb{R}$ and $h \in H^{-1/2}(\Gamma_3)^3$.

In addition, p_0 must be *defined up to an additive constant*. The restriction of p_0 to the ith connected component of Γ_2 will be denoted by p_{0i}, i.e.

$$p_{0i} = p_0|_{\Gamma_{2i}}, \quad i = 1, \ldots, r. \tag{7}$$

2.2. Functional framework

We assume that the reader is somewhat familiar with the space of square integrable functions L^2, the Sobolev space H^1 of square integrable functions with square integrable derivatives and its subspace H_0^1 of functions with zero restrictions on boundaries. As usual $\mathscr{C}_0^p$ denotes the space of p-continuously differentiable functions with compact support. Let us introduce the functional spaces

$$V = \{\mathbf{v} \in H^1(\Omega)^3 | \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} = 0 \text{ on } \Gamma_1,\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_2, \mathbf{v} \cdot n = 0 \text{ on } \Gamma_3\}, \tag{8}$$

$$V_0 = \left\{\mathbf{v}_0 \in V \middle| \int_{\Gamma_{2i}} \mathbf{v}_0 \cdot \mathbf{n} ds = 0 \quad \forall i = 1, \ldots, r\right\}. \tag{9}$$

The non-homogeneous analogues are defined for any function $\mathbf{w}$ of $H^1(\Omega)^3$ by

$$V(\mathbf{w}) = \{\mathbf{v} \in H^1(\Omega)^3 | \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega,\ \mathbf{v} - \mathbf{w}|_{\Gamma_1} = 0,\ \mathbf{v} - \mathbf{w} \times \mathbf{n}|_{\Gamma_2} = 0,\ \mathbf{v} - \mathbf{w} \cdot \mathbf{n}|_{\Gamma_3} = 0\} \tag{10}$$

and for any vector $\mathbf{F} = \{F_i\}_{i=1}^r$ such that $\sum_{i=1}^r F_i = 0$ by

$$V_0(\mathbf{w}, \mathbf{F}) = \left\{\mathbf{v}_0 \in V(\mathbf{w}) \middle| \int_{\Gamma_{2i}} \mathbf{v}_0 \cdot \mathbf{n} ds = F_i\right\}. \tag{11}$$

Let $a(\cdot,\cdot)$ be the bilinear continuous form

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx \quad \forall \mathbf{u}, \mathbf{v} \in H^1(\Omega)^3. \tag{12}$$

2.3. Variational formulation of the problem: existence and uniqueness results

Let us consider the variational problem

$$\text{find } \mathbf{u} \in H^1(\Omega)^3 \text{ such that} \tag{13a}$$

$$\mathbf{u} - \mathbf{U}_0 \in V, \tag{13b}$$

$$a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{13c}$$

where

$$L(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx + \nu \int_{\Gamma_3} (\mathbf{h} \times \mathbf{n}) \cdot \mathbf{v} ds - \int_{\Gamma_2} p_0 \mathbf{v} \cdot \mathbf{n} ds \quad \forall \mathbf{v} \in H^1(\Omega)^3. \tag{14}$$

Remark 1

It is not necessary to know $\mathbf{U}_0$ explicitly, because condition (13b) can be replaced by $\mathbf{u} \in V(\mathbf{U}_0)$. □

Since the bilinear form $a(\cdot, \cdot)$ is V-elliptic and since $L(\cdot)$ is a linear form which is continuous in V, the Lax–Milgram lemma yields the following.

Theorem 1

The variational problem (13) has one and only one solution. □

Remark 2

If Γ_1 is empty, then the bilinear form may not be V-elliptic. This case has been studied in Reference 9. □

2.4. Equivalence between the boundary value problem and the variational problem

To establish the equivalence between problems (1), (3) and (13), we shall begin by proving the following.

Theorem 2

If $\mathbf{u} \in \mathscr{C}^2(\bar{\Omega})$ and $p \in \mathscr{C}^1(\bar{\Omega})$ are classical solutions of the boundary value problem (1), (3), then $\mathbf{u}$ is a solution of the variational problem (13).

Proof. This is done in the usual way. Multiplying equation (1a) by $\mathbf{v}$ in V, integrating by parts in Ω and using (1b) and the identity

$$-\Delta\mathbf{u} = \nabla \times (\nabla \times \mathbf{u}) - \nabla(\nabla \cdot \mathbf{u}), \tag{15}$$

we obtain

$$\nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v} \mathrm{d}x + \nu \int_\Gamma [(\nabla \times \mathbf{u}) \times \mathbf{n}] \cdot \mathbf{v} \mathrm{d}s - \int_\Gamma p\mathbf{v} \cdot \mathbf{n} \mathrm{d}s,$$

which, using the fact that $p = p_0$ on Γ_2 and $(\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$ on Γ_3, implies that $\mathbf{u}$ verifies (13c). Furthermore, using the fact that $\mathbf{u}$ is of class $\mathscr{C}^2(\bar{\Omega})$ (which implies that $\mathbf{u} \in H^1(\Omega)^3$), we deduce from (1b), (3) and (6) that $(\mathbf{u} - \mathbf{U}_0)$ belongs to V. Thus $\mathbf{u}$ is a solution of (13). Theorem 2 is therefore proved. □

Reciprocally, we have the following.

Theorem 3

Let $\mathbf{u}$ be a solution of the variational problem (13). Then it has enough regularity* for $(\nabla \times \mathbf{u})|_\Gamma$ to be well defined and there exists a smooth function p† defined up to a constant such that $\mathbf{u}$ and p are solutions of the boundary value problem (1), (3) in the distribution sense.

Proof. Let $\mathbf{u}$ be a solution of problem (13). The fact that $\mathbf{u} - \mathbf{U}_0$ belongs to V and that $\mathbf{U}_0$ verifies (6) implies that $\mathbf{u}$ satisfies equation (1b) in the sense of distributions in Ω and it also implies that $\mathbf{u}$ verifies (3a) and the first parts of the boundary conditions (3b) and (3c) in the sense of the traces of functions of $H^1(\Omega)^3$.

Now let us take a smooth divergence-free function $\mathbf{v}$ as test function in (13c). Using the definition of a distribution derivative, we have

$$\int_\Omega [\nu\nabla \times (\nabla \times \mathbf{u})] \cdot \mathbf{v} \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v} \mathrm{d}x \quad \forall \mathbf{v} \in \mathscr{C}_0^\infty(\Omega)^3; \qquad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega.$$

* It belongs to $H(\Delta, \Omega)^3$, where $H(\Delta, \Omega) \equiv \{q \in L^2(\Omega) \mid \Delta q \in L^2(\Omega)\}$; the functions of $H(\Delta, \Omega)$ have traces on Γ which belong to $H^{-1/2}(\Gamma)$.

† $p \in H(\Delta, \Omega)\mathbb{R}$.

Since the boundary Γ of Ω is locally Lipschitz, it follows from Theorem I.2.3 on p. 25 of Reference 10 (see also Reference 3, Remark I.1.9, p. 19) that there exists $p \in L^2(\Omega)$ defined up to a constant such that

$$\nu\nabla \times (\nabla \times \mathbf{u}) + \nabla p = \mathbf{f} \tag{16}$$

in the sense of distributions. This proves (1a). Moreover, applying the divergence operator to equation (16), we obtain

$$\Delta p = \nabla \cdot \mathbf{f}. \tag{17}$$

On the other hand, applying the rotational operator on both sides of (1a), we have

$$\nu\Delta(\nabla \times \mathbf{u}) = -\nabla \times \mathbf{f}. \tag{18}$$

However, $\mathbf{f}$ being smooth enough (see footnote on p. 269;) then (17) and (18) imply that p and $\nabla \times \mathbf{u}$ are smooth in the sense of the footnote on p. 271. Moreover, multiplying (16) by $\mathbf{v}$ in V, integrating by parts in Ω and using (13c), we are led to

$$\int_\Omega [\nu\nabla \times (\nabla \times \mathbf{u}) + \nabla p] \cdot \mathbf{v} dx - \nu\int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx = -\nu\int_{\Gamma_3} (\mathbf{h} \times \mathbf{n}) \cdot \mathbf{v} ds + \int_{\Gamma_2} p_0 \mathbf{v} \cdot \mathbf{n} ds \tag{19}$$

for all $\mathbf{v} \in V$.

The second parts of the boundary conditions (3b) and (3c) are implicitly contained in (19). To interpret them, we must be able to integrate by parts the first term on the left-hand side of (19). This is not possible *a priori* because of the lack of regularity of $\mathbf{u}$ and p. However, if we assume that $\mathbf{u}$ is smooth,* then using the regularity of $\mathbf{f}$ (see footnote on p. 269) and (16), we have p in $H^1(\Omega)$ and we can therefore integrate this term by parts. We obtain

$$-\nu\int_{\Gamma_3} [(\nabla \times \mathbf{u}) \times \mathbf{n}] \cdot \mathbf{v} ds + \int_{\Gamma_2} p\mathbf{v} \cdot \mathbf{n} ds = -\nu\int_{\Gamma_3} (\mathbf{h} \times \mathbf{n}) \cdot \mathbf{v} ds + \int_{\Gamma_2} p_0\mathbf{v} \cdot \mathbf{n} ds \quad \forall \mathbf{v} \in V.$$

Taking in this expression test functions which are zero on Γ_1 and Γ_2, we get (3c). Next we get (3b) by taking test functions which are zero on Γ_1 and Γ_3 and using the fact that $\int_{\Gamma_2} \mathbf{v} \cdot \mathbf{n} ds = 0$ for all $\mathbf{v} \in V$. This completes the proof of Theorem 3. □

2.5. Computation of the fluxes of the velocity on the connected components of Γ_2

It can be shown[9] that there exists $r-1$ functions of V satisfying

$$\int_{\Gamma_{2j}} \mathbf{z}_i \cdot \mathbf{n} ds = \delta_{ij}, \quad j = 1, \ldots, r-1, \qquad \int_{\Gamma_{2r}} \mathbf{z}_i \cdot \mathbf{n} ds = -1.$$

Then for $i = 1, \ldots, r-1$ let ω_{0i} be the solution in V_0 of

$$a(\omega_{0i}, \mathbf{v}_0) = -a(z_i, \mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0.$$

Finally define

$$\omega_i = \omega_{0i} + z_i. \tag{20}$$

It can be shown that $\{\omega_i\}_{i=1}^{r-1}$ form a basis of the space of functions $\mathbf{v} \in V$ which satisfy $a(\mathbf{v}, \mathbf{v}_0) = 0$ for all $\mathbf{v}_0 \in V_0$.

In what follows we shall denote by F_{2j}^0 the flux of $\mathbf{U}_0$ through Γ_{2j}, i.e.

$$F_{2j}^0 = \int_{\Gamma_{2j}} \mathbf{U}_0 \cdot \mathbf{n} ds, \quad j = 1, \ldots, r. \tag{21}$$

* For instance, if $\nabla \times (\nabla \times \mathbf{u}) \in L^2(\Omega)^3$.

Proposition 1

If $\mathbf{u}$ is a solution of the variational problem (13) and if F_{2j} is the flux of $\mathbf{u}$ through Γ_{2j}, which is defined by

$$F_{2j} = \int_{\Gamma_{2j}} \mathbf{u} \cdot \mathbf{n} ds, \quad j = 1, \ldots, r, \tag{22a}$$

then

$$\sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) a(\omega_i, \omega_j) = L(\omega_j) - a(\mathbf{U}_0, \omega_i) \quad \forall i = 1, \ldots, r-1. \tag{22b}$$

Proof. Taking $\mathbf{v} = \omega_i$ in (13c), we have

$$a(\mathbf{u}, \omega_i) = L(\omega_i),$$

which implies

$$a(\mathbf{u} - \mathbf{U}_0, \omega_i) = L(\omega_i) - a(\mathbf{U}_0, \omega_i).$$

On the other hand, since the ω_i form a basis,

$$\mathbf{u} - \mathbf{U}_0 = \mathbf{v}_0 + \sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) \omega_j,$$

with $\mathbf{v}_0 \in V_0$. Since $a(\mathbf{v}_0, \omega_i) = 0$, upon combining the two identities, we get (22). Proposition 1 is thus proved. □

It is easy to verify that the coefficients $a(\omega_i, \omega_j)$ which intervene in formula (22b) define a symmetric matrix that is positive definite. Thus this formula allows the direct computation of the fluxes $F_{21}, \ldots, F_{2(r-1)}$ of $\mathbf{u}$ through the first $r - 1$ components of Γ_2. It can be noticed that the flux F_{2r} of $\mathbf{u}$ through Γ_{2r} is completely determined by the fluxes $F_{21}, \ldots, F_{2(r-1)}$, the incompressibility condition (1b) and the boundary conditions (3). Indeed, we have

$$F_{2r} = -\left(\int_{\Gamma_1} \mathbf{u}_0 \cdot \mathbf{n} ds + \sum_{j=1}^{r-1} F_{2j} + \int_{\Gamma_3} \mathbf{b} \cdot \mathbf{n} ds \right). \tag{23}$$

2.6. The Stokes equations with prescribed fluxes on the connected components of Γ_2

Let us consider the following variant of the boundary problem (1), (3): find functions $\mathbf{u}$, p and constants $C_1, \ldots, C_r$ defined up to an additive constant (i.e. we are looking for the differences $C_i - C_r$ for $i = 1, \ldots, r - 1$) such that

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1, \tag{24a}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{on } \Gamma_2, \tag{24b}$$

$$p = \bar{p}_{0i} + C_i \quad \text{on } \Gamma_{2i} \quad \forall i = 1, \ldots, r, \tag{24c}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{on } \Gamma_3, \tag{24d}$$

$$(\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3, \tag{24e}$$

$$\int_{\Gamma_{2j}} \mathbf{u} \cdot \mathbf{n} ds = F_{2j} \quad \forall j = 1, \ldots, r, \tag{24f}$$

where ν, $\mathbf{f}$, $\mathbf{u}_0$, $\mathbf{a}$, $\bar{p}_0$, $\mathbf{b}$, $\mathbf{h}$ and the fluxes $F_{21}, \ldots, F_{2r}$ are given. It can be remarked that the only difference between this new boundary-value problem and the boundary-value problem (1), (3) is the fact that while in (1), (24) the fluxes are included within the data and the constants C_i are unknowns, in problem (1), (3) we have the opposite situation.

Integrating (1b) by parts in Ω, we see that a necessary condition for (1), (24) to possess a solution is that the global flux through Γ be zero, i.e.

$$\int_{\Gamma_1} \mathbf{U}_0 \cdot \mathbf{n} ds + \sum_{j=1}^{r} F_{2j} + \int_{\Gamma_3} \mathbf{b} \cdot \mathbf{n} ds = 0, \tag{25}$$

which we will assume fulfilled. Indeed, we will suppose that the prescribed fluxes are $F_1, \ldots, F_{r-1}$ and that F_{2r} is given in such a way that (25) holds.

The formulation of the problem is

$$\text{find } \mathbf{u} \in V_0(\mathbf{U}_0, \mathbf{F}) \text{ such that}$$
$$a(\mathbf{u}, \mathbf{v}_0) = \bar{L}(\mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0.$$

Notice again that it is not necessary to know $\mathbf{U}_0$ explicitly.

For theoretical purposes it is more convenient to work with the following equivalent formulation:

find $\mathbf{u} \in H^1(\Omega)^3$ and constants $C_1, \ldots, C_r$ (defined up to an additive constant) such that (26a)

$$\mathbf{u} - \mathbf{U}_1 \in V_0, \tag{26b}$$

$$a(\mathbf{u}, \mathbf{v}_0) = \bar{L}(\mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0, \tag{26c}$$

$$C_i - C_r = \bar{L}(\omega_i) - a(\mathbf{U}_0, \omega_i) - \sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) a(\omega_i, \omega_j) \quad \forall i = 1, \ldots, r-1, \tag{26d}$$

where V_0 is defined by (9), $\bar{L}(\cdot)$ is defined by

$$\bar{L}(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx + \nu \int_{\Gamma_3} (\mathbf{h} \times \mathbf{n}) \cdot \mathbf{v} ds - \int_{\Gamma_2} \bar{p}_0 \mathbf{v} \cdot \mathbf{n} ds \quad \forall \mathbf{v} \in H^1(\Omega)^3 \tag{27}$$

and $\mathbf{U}_1 \in H^1(\Omega)^3$ is defined by

$$\mathbf{U}_1 = \mathbf{U}_0 + \sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) \omega_j. \tag{28}$$

Theorem 4

The variational problem (26) has one and only one solution. Its interpretation is none other than the boundary-value problem (1), (24). □

2.7. Computation of the normal derivative of the pressure on Γ_3.

In this subsection we will go back to problem (1), (3). We shall prove that for regular boundaries* the normal derivative of the pressure on Γ_3 can be explicitly calculated from the data of problem (1), (3).

* Γ of class $\mathscr{C}^{1,1}$.

We denote by X the space of smooth functions with zero values on Γ_1 and Γ_2, defined by

$$X = \{\phi \in H^2(\Omega) | \phi = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}. \tag{29}$$

Let $\mathbf{u}$ be a solution of the variational problem (13) and let p be the corresponding pressure. According to what was seen in the proof of Theorem 3, it is clear that $\mathbf{u}$ and p satisfy the equations

$$\nu\nabla \times (\nabla \times \mathbf{u}) + \nabla p = \mathbf{f}, \tag{30a}$$

$$\Delta p = \nabla \cdot \mathbf{f} \tag{30b}$$

in the sense of distributions in Ω. Therefore, if $\phi \in X$, multiplying (30a) by $\Delta\phi$ and (30b) by ϕ, we obtain

$$\int_\Omega [\nu\nabla \times (\nabla \times \mathbf{u}) + \nabla p] \cdot \nabla\phi \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \nabla\phi \mathrm{d}x \quad \forall\phi \in X, \tag{31a}$$

$$\int_\Omega \Delta p \; \phi \mathrm{d}x = \int_\Omega \nabla \cdot \mathbf{f} \; \phi \mathrm{d}x \quad \forall\phi \in X. \tag{31b}$$

The identities (31) implicitly contain the property of p that is of our interest. To interpret it, we need to integrate by parts the left-hand-side terms of (31a) and (31b). To this end we shall assume that $\nabla \times (\nabla \times \mathbf{u})$ is square integrable. This implies that ∇p is also square integrable and we can therefore use Green's formula in the left-hand-side terms of (31). We obtain

$$-\nu \int_\Gamma [(\nabla \times \mathbf{u}) \times \mathbf{n}] \cdot \nabla\phi \mathrm{d}s + \int_\Omega \nabla p \cdot \nabla\phi \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \nabla\phi \mathrm{d}x \quad \forall\phi \in X, \tag{32a}$$

$$-\int_\Omega \nabla p \cdot \nabla\phi + \int_\Gamma \frac{\partial p}{\partial n} \phi \mathrm{d}s = -\int_\Omega \mathbf{f} \cdot \nabla\phi + \int_\Gamma \mathbf{f} \cdot \mathbf{n}\phi \mathrm{d}s \quad \forall\phi \in X. \tag{32b}$$

Then

$$\int_\Gamma \frac{\partial p}{\partial n} \phi \mathrm{d}s = \int_\Gamma \mathbf{f} \cdot \mathbf{n}\phi \mathrm{d}s + \nu \int_\Gamma [(\nabla \times \mathbf{u}) \times \mathbf{n}] \cdot \nabla\phi \mathrm{d}s \quad \forall\phi \in X. \tag{33}$$

Definition 1

Let $\mathbf{g}$ be a vector-valued function on Γ that verifies $\mathbf{g} \cdot \mathbf{n}|_\Gamma = 0$. We shall call the *tangential divergence* of $\mathbf{g}$ the distribution $\nabla_t \cdot \mathbf{g}$ defined by*

$$\int_\Gamma \nabla_t \cdot \mathbf{g}\varphi ds = -\int_\Gamma \mathbf{g} \cdot \nabla\phi \mathrm{d}s \quad \forall\varphi, \tag{34}$$

where $\phi \in H^2(\Omega)$ is any extension of φ to Ω. □

Remark 3

For a smooth function $\mathbf{g}$ defined in $\bar{\Omega}$ with values in $\mathbb{R}^3$ and satisfying $\mathbf{g} \cdot \mathbf{n}|_\Gamma = 0$, an explicit computation (see e.g. Reference 11, Lemma 4.9 or Reference 12, Section 4) shows that

$$\nabla_t \cdot \mathbf{g} = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} - \sum_{i,j=1}^{3} \frac{\partial g_i}{\partial x_j} n_i n_j \quad \text{on } \Gamma. \qquad \square \tag{35}$$

* Precisely, $\mathbf{g}$ is a distribution of $H^{-1/2}(\Gamma)^3$ with $\nabla_t \cdot g \in H^{-3/2}(\Gamma)$ and the variational equation is for all $\phi \in H^{3/2}(\Gamma)$.

Going back to (33), it follows that

$$\int_\Gamma \frac{\partial p}{\partial n}\,\phi \mathrm{d}s = \int_\Gamma \mathbf{f}\cdot\mathbf{n}\phi \mathrm{d}s - \nu\int_\Gamma \nabla_t\cdot[(\nabla\times\mathbf{u})\times\mathbf{n}]\phi\mathrm{d}s \quad \forall\phi\in X. \tag{36}$$

Now let us consider a smooth† function ϕ whose restriction to Γ is zero everywhere except on Γ_3. The boundary condition (3c) says that $(\nabla\times\mathbf{u})\times\mathbf{n} = \mathbf{h}\times\mathbf{n}$ on Γ_3 and it then follows from (36) that

$$\int_{\Gamma_3} \frac{\partial p}{\partial n}\,\phi \mathrm{d}s = \int_{\Gamma_3} \mathbf{f}\cdot\mathbf{n}\phi \mathrm{d}s - \nu\int_{\Gamma_3} \nabla_t\cdot(\mathbf{h}\times\mathbf{n})\phi\mathrm{d}s. \tag{37}$$

Thus we have proved the following.

Proposition 2

If $\mathbf{u}$ is a smooth solution of the variational problem (13), then the pressure p associated with $\mathbf{u}$ satisfies the boundary condition*

$$\frac{\partial p}{\partial n} = \mathbf{f}\cdot\mathbf{n} - \nu\nabla_t\cdot(\mathbf{h}\times\mathbf{n}) \quad \text{on } \Gamma_3. \qquad \square \tag{38}$$

3. THE NAVIER–STOKES EQUATIONS WITH BOUNDARY CONDITIONS INVOLVING THE PRESSURE

In this section we will extend some of the results of Section 2 to the case of the Navier–Stokes equations. We will successively study two versions of the Navier–Stokes problem associated with (1), (3) (first, with homogeneous boundary conditions on the velocity; secondly, with non-homogeneous conditions).

3.1. Description of the Navier–Stokes problem

In this subsection we are interested in studying the (stationary) Navier–Stokes equations (2) with the boundary conditions (4). To study problem (2), (4), we consider its variational formulation

$$\text{find } \mathbf{u}\in H^1(\Omega)^3 \text{ such that} \tag{39a}$$

$$\mathbf{u} - \mathbf{U}_0 \in V, \tag{39b}$$

$$a(\mathbf{u},\ \mathbf{v}) + b(\mathbf{u},\ \mathbf{u},\ \mathbf{v}) = L(\mathbf{v}) \quad \forall\mathbf{v}\in V, \tag{39c}$$

where the trilinear form $b(\cdot,\cdot,\cdot)$, is defined by

$$b(\cdot,\ \cdot,\ \cdot)\ :\ H^1(\Omega)^3\times H^1(\Omega)^3\times H^1(\Omega)^3 \to \mathbb{R}, \tag{40a}$$

$$b(\mathbf{u},\ \mathbf{v},\ \mathbf{w}) = \int_\Omega [(\nabla\times\mathbf{u})\times\mathbf{v}]\cdot\mathbf{w}\mathrm{dx} \quad \forall\mathbf{u},\ \mathbf{v},\ \mathbf{w}\in H^1(\Omega)^3. \tag{40b}$$

Theorem 5

There exists ν^*, a function of the data of the problem, such that if $\nu > \nu^*$, then problem (39) has one and only one solution. $\square$

† $\phi\in H^2(\Omega)$.
* As elements of $H^{-3/2}(\Gamma_3)$.

Theorem 6

If $\mathbf{u}$ in $\mathscr{C}^2(\bar{\Omega})$ and p in $\mathscr{C}^1(\bar{\Omega})$ are classical solutions of the boundary-value problem (2), (4), then $\mathbf{u}$ is a solution of the variational problem (39). □

Reciprocally, we have the following.

Theorem 7

Let $\mathbf{u}$ be a solution of the variational problem (39). Then there exists $p \in L^2(\Omega)$ defined up to a constant such that $\mathbf{u}$ and p are solutions of the boundary-value problem (2), (4) in the sense of distributions.

Proof. Let $\mathbf{u}$ be a solution of problem (39). First note that (39a) and (39b) imply (2b) and the Dirichlet boundary conditions on $\mathbf{u}$. On the other hand, taking divergence-free functions in $\mathscr{C}_0^\infty(\Omega)^3$ as test functions in (39) and using the definition of distribution derivative, we conclude that

$$\int_\Gamma \nu\nabla \times (\nabla \times \mathbf{u}) + [(\nabla \times \mathbf{u}) \times \mathbf{u}] \cdot \mathbf{v}dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}dx \quad \forall \mathbf{v} \in \mathscr{C}_0^\infty(\Omega)^3, \qquad \nabla \cdot \mathbf{v} = 0,$$

which implies that there exists a q in $L^2(\Omega)$ defined up to a constant such that (see Reference 10, Theorem I.2.3, p. 25)

$$\nu\nabla \times (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla q = \mathbf{f}.$$

Then we put

$$p = q - \tfrac{1}{2}|\mathbf{u}|^2$$

and thus we have

$$\nu\nabla \times (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u} + \tfrac{1}{2}\nabla|\mathbf{u}|^2 + \nabla p = \mathbf{f}. \tag{41}$$

However, $\nabla \cdot \mathbf{u} = 0$, so we deduce from (15) that

$$-\nu\nabla\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \tag{42}$$

in the sense of distributions in Ω. Moreover, multiplying (41) by a test function $\mathbf{v}$ in V and using (39c), we obtain by difference

$$\int_\Omega \left[\nu\nabla \times (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla(p + \tfrac{1}{2}|\mathbf{u}|^2)\right] \cdot \mathbf{v}dx - \nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})dx$$
$$- \int_\Omega [(\nabla \times \mathbf{u}) \times \mathbf{u}] \cdot \mathbf{v}dx = -\nu \int_{\Gamma_3} [\mathbf{h} \times \mathbf{n}] \cdot \mathbf{v}ds + \int_{\Gamma_2} p_0\mathbf{v} \cdot \mathbf{n}ds \quad \forall \mathbf{v} \in V. \tag{43}$$

As in the linear case, the identity (43) implicitly contains the second parts of the boundary conditions (4b) and (4c). If $\nabla \times (\nabla \times \mathbf{u})$ and $(\nabla \times \mathbf{u}) \times \mathbf{u}$ are square integrable, we deduce from (41) that $(p + \frac{1}{2}|\mathbf{u}|^2)$ belongs to $H^1(\Omega)$, and integrating by parts the left-hand side of (43), we obtain

$$-\nu \int_{\Gamma_3} [(\nabla \times \mathbf{u}) \times \mathbf{n}] \cdot \mathbf{v}ds + \int_{\Gamma_2} \left(p + \tfrac{1}{2}|\mathbf{u}|^2\right)\mathbf{v} \cdot \mathbf{n}ds = -\nu \int_{\Gamma_3} (\mathbf{h} \times \mathbf{n}) \cdot \mathbf{v}ds + \int_{\Gamma_2} p_0\mathbf{v} \cdot \mathbf{n}ds \quad \forall \mathbf{v} \in V.$$

Taking in this identity test functions $\mathbf{v}$ with compact support in Γ_3, we get (4c). Next, (4b) follows easily if we take test functions with compact support in Γ_2 and we use the fact that the normal trace of the functions of V have zero mean value on Γ_2. Theorem 7 is therefore proved. □

3.2. The Navier–Stokes equations with prescribed fluxes on the connected components of Γ_2

As for the linear case, we will also study the following variant of the Navier–Stokes problem (2), (4): find $\mathbf{u}$, p and constants $C_1, \ldots, C_r$ defined up to an additive constant (i.e. we are looking for the differences $C_i - C_r$ for $i = 1, \ldots, r-1$) such that

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1, \tag{44a}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{on } \Gamma_2, \tag{44b}$$

$$p + \tfrac{1}{2}|\mathbf{u}|^2 = \bar{p}_{0i} + C_i \quad \text{on } \Gamma_{2i} \quad \forall i = 1, \ldots, r, \tag{44c}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{on } \Gamma_3, \tag{44d}$$

$$(\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3, \tag{44e}$$

$$\int_{\Gamma_{2j}} \mathbf{u} \cdot \mathbf{n}\, ds = F_{2j} \quad \forall j = 1, \ldots, r, \tag{44f}$$

where v, $\mathbf{f}$, $\mathbf{u}_0$, $\mathbf{a}$, $\bar{p}_0$, $\mathbf{b}$, $\mathbf{h}$ and the fluxes $F_{21}, \ldots, F_{2r}$ are the problem data.

Setting

$$p_{0i} = \bar{p}_{0i} + C_i, \quad i = 1, \ldots, r, \tag{45}$$

and using Theorem 6, it is a straightforward matter to show that if $\mathbf{u} \in \mathscr{C}^2(\bar{\Omega})$, $p \in \mathscr{C}^1(\bar{\Omega})$ and $C_1, \ldots, C_r$ are classical solutions of problems (2), (44), then $\mathbf{u}$ and the constants $C_1, \ldots, C_r$ are solutions of the variational problem

find $\mathbf{u} \in H^1(\Omega)^3$ and constants $C_1, \ldots, C_r$ (defined up to an additive constant) such that (46a)

$$\mathbf{u} \in V_0(\mathbf{U}_0, \mathbf{F}), \tag{46d}$$

$$a(\mathbf{u}, \mathbf{v}_0) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_0) = \bar{L}(\mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0. \tag{46c}$$

Reciprocally, if $\mathbf{u}$ and $C_1, \ldots, C_r$ are solutions of (46), defining p_0 by (45), it is easy to check that $\mathbf{u}$ is a solution of the variational problem (39). According to Theorem 7, there then exists $p \in L^2(\Omega)$ defined up to a constant such that $\mathbf{u}$ is a solution of (2), (44). On the other hand, $\mathbf{u}$ satisfies the flux conditions (44f), since $\mathbf{u} - \mathbf{U}_1$ belongs to V_0 and condition (25) is verified.

This proves that (46) is a variational formulation for the boundary-value problem (2), (44). As regards the existence and uniqueness of a solution of this problem, we can prove that if either v is sufficiently large relative to the data or the data are small enough with respect to v, then problem (46) admits one and only one solution.

3.3. Computation of the normal derivative of the pressure on Γ_3

In this subsection we aim at extending formula (38) (Proposition 2) to the non-linear case. In contrast with the Stokes case, for the Navier–Stokes equations this formula depends on the solution as well as on the problem data.

Proposition 3

If $\mathbf{u}$ is any smooth* solution of the variational problem (39), then $p + \frac{1}{2}|\mathbf{u}|^2$ satisfies the following boundary condition on Γ_3,

$$\frac{\partial}{\partial n}(p + \tfrac{1}{2}|\mathbf{u}|^2) = [\mathbf{f} - (\nabla \times \mathbf{u}) \times \mathbf{u}] \cdot \mathbf{n} - \nu \nabla_t \cdot (\mathbf{h} \times \mathbf{n}), \tag{47}$$

in the sense of distributions,† where $\nabla_t \cdot (\mathbf{h} \times \mathbf{n})$ denotes the tangential derivative of $\mathbf{h} \times \mathbf{n}$ on Γ_3 (see Definition 1).

Proof. Let $\mathbf{u}$ be a smooth solution of problem (39) and let p be the corresponding pressure. According to Theorem 7, $\mathbf{u}$ and p satisfy

$$\nu \nabla \times (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla(p + \tfrac{1}{2}|\mathbf{u}|^2) = \mathbf{f}. \tag{48a}$$

Thus, applying the divergence operator on both sides of this equation, we have

$$\Delta\left(p + \tfrac{1}{2}|\mathbf{u}|^2\right) = \nabla \cdot [\mathbf{f} - (\nabla \times \mathbf{u}) \times \mathbf{u}]. \tag{48b}$$

From this it follows that

$$\int_\Omega [\nu \nabla \times (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla(p + \tfrac{1}{2}|\mathbf{u}|^2)] \cdot \nabla\phi = \int_\Omega \mathbf{f} \cdot \nabla\phi \qquad \forall \phi \in X, \tag{49a}$$

$$\int_\Omega \Delta(p + \tfrac{1}{2}|\mathbf{u}|^2)\phi = \int_\Omega \nabla \cdot [\mathbf{f} - (\nabla \times \mathbf{u}) \times \mathbf{u}]\phi \quad \forall \phi \in X, \tag{49b}$$

where X is the space defined by (29). Now we can follow step by step the proof of the linear case, but replacing $\mathbf{f}$ by $\mathbf{f} - (\nabla \times \mathbf{u}) \times \mathbf{u}$ and p by $p + \frac{1}{2}|\mathbf{u}|^2$. This completes the proof of Proposition 3. □

4. NUMERICAL RESULTS

The aim of this section is to present some numerical experiments concerning the approximation of one of the Navier–Stokes boundary-value problems which have been studied in Section 3. These numerical results are relative to flows in a network of pipes and flow around an obstacle in a pipe. In both cases we have $\Gamma_3 = \varnothing$, $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{a} = \mathbf{0}$ (i.e. homogeneous boundary conditions on the velocity).

4.1. Two-dimensional results

4.1.1. Brief description of the numerical method. The numerical method which we use as an operator-splitting method with a θ-scheme as proposed and developed by Glowinski for the Navier–Stokes problem (see Reference 13, Chapter VII). Therefore at each step of the algorithm we solve first a Stokes-type problem, next a non-linear convective problem and then again a Stokes-type problem. This scheme has a time truncation error of $O(|\Delta t|^2)$ and appears to be unconditionally stable. The non-linear problem is solved at each time step by a least squares conjugate gradient method. On the other hand, to solve the Stokes-type problems, we use a variant of the splitting method of Glowinski and

* $\nabla \times (\nabla \times \mathbf{u}) \in L^2(\Omega)^3$, $(\nabla \times \mathbf{u}) \times \mathbf{u} \in L^2(\Omega)^3$ and $\nabla \cdot [(\nabla \times \mathbf{u}) \times \mathbf{u}] \in L^2(\Omega)$.

† In $H^{-3/2}(\Gamma_3)$.

Pironneau[14] (see also Reference 13, Chapter VII; for variants of this method, including the case of boundary conditions involving the pressure, see Reference 15).

4.1.2. Discretization in space. Next we consider the space approximation of the time-dependent problems associated with (1), (3) and (2), (4). For simplicity we suppose that Ω is a bounded polygonal domain of R^2 and we introduce a classical triangular τ_h *of* Ω consisting of a finite set of triangles. We define a new triangulation $\tau_{h/2}$ from τ_h by subdividing each triangle $T \in \tau_h$ into four subtriangles (by joining the mid-sides of T). The pressure is approximated by piecewise polynomial functions which are continuous on each triangle of τ_h. To approximate the velocity, we also use Lagrange finite elements. The discrete velocities are continuous functions which when restricted to each subtriangle $T \in \tau_h$ are polynomials of degree one.

4.1.3. Numerical results

Flow in a network of pipes

Let us consider a bidimensional network of pipes whose shape looks like a T (such a network will be called a T-shaped pipe network). The lateral surfaces of the pipes constitute the portion Γ_1. Γ_2 has three connected components: the inflow section Γ_{21}, which is located at the bottom of the T, and two outflow sections Γ_{22} and Γ_{23}, which are located at the two extremes of the horizontal branch of the T. The Navier–Stokes equations were solved numerically with the data

$$\nu = 0.025, \qquad \mathbf{f} = \mathbf{0}, \qquad p_0 = C_i \quad \text{on } \Gamma_{2i}, \quad i = 1,\ 2,\ 3.$$

Figure 1 shows the computed solution correspondong to the case where the pressure differences are given by the values

$$C_1 - C_2 = C_1 - C_3 = 2.$$

We have drawn the velocity field as well as the isobaric and isorotational lines. The flow is naturally symmetric. Figure 2 shows another numerical result in which the symmetry on the data was broken by setting

$$C_1 - C_2 = 4 > C_1 - C_3 = 2.$$

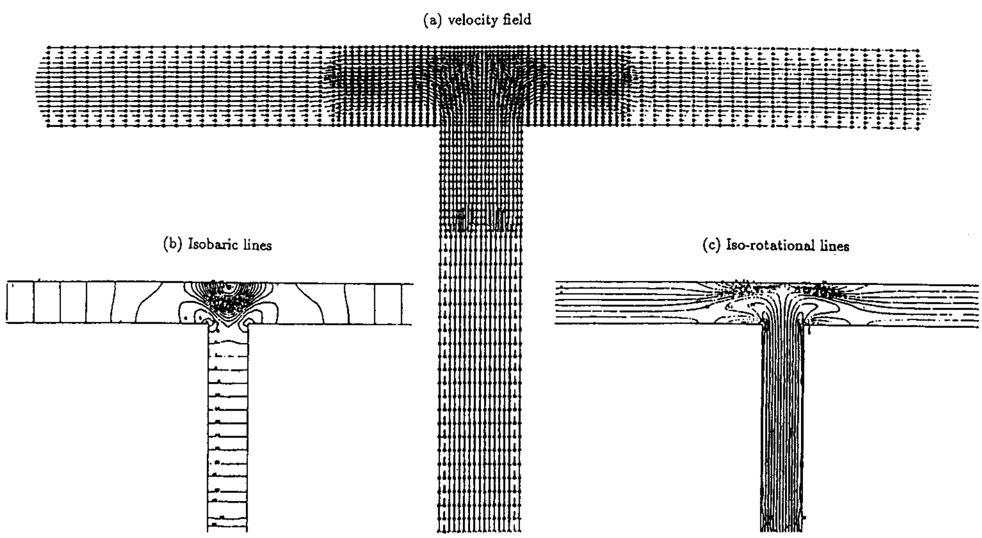

Figure 1. Velocity in a two-dimensional T-shaped bifurcation at $Re = 40$ when the pressure is imposed on the inlet and outlet boundaries; $C_1 - C_2 = C_1 - C_3 = 2$ (computed by C. Bègue)

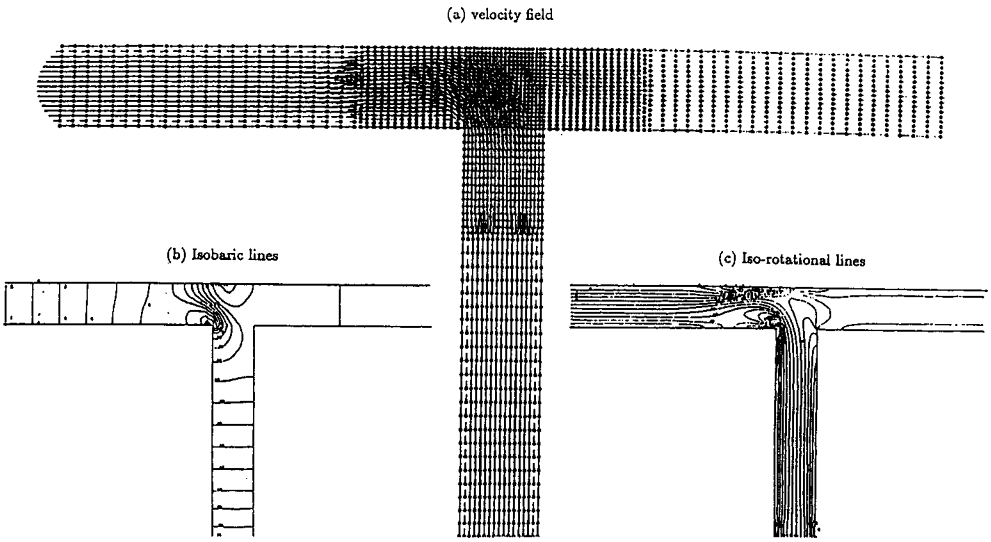

Figure 2. Same as Figure 1 but with $C_1 - C_2 = 4 > C_1 - C_3 = 2$ (computed by C. Bègue)

Flow in a pipe around an obstacle

In this second example the flow take place in the straight section between two parallel plates. Equidistant from both there is a cylindrical pipe with generating lines parallel to the plates. Γ_1 consists of the boundary of the sections of the pipe and of the plates. Γ_2 has two connected components: the inflow part Γ_{21} and the outflow part Γ_{22}. The bidimensional Navier–Stokes equations were solved with the data

$$\nu = 0.025, \qquad \mathbf{f} = \mathbf{0}, \qquad p_0 = C_1 \quad \text{on } \Gamma_{21}, \qquad p_0 = C_2 \quad \text{on } \Gamma_{22},$$

where the C_i are constant. Setting $C = C_1 - C_2$, Figure 3 shows the computed solutions corresponding to the three values

$$C = 1, \qquad C = 2, \qquad C = 3.$$

In each case we have drawn the velocity field and the isorotational lines. We can observe that as the pressure difference C increases, the two vortices behind the cylinder get longer and the profile of the outflow velocity becomes more and more perturbed by the presence of the pipe. In particular, it becomes very different from the parabolic profiles obtained in a Poiseuille flow without any obstacle. Figure 4 shows the value of the flux $F_{22} = -F_{21} = \int_{\Gamma_{22}} \mathbf{u} \cdot \mathbf{n}\, ds$ as a function of C for C between 0 and 3·5.

4.2. *Three-dimensional results*

4.2.1. Physiological flows. Applications of imposed pressure boundary conditions may be found in biomechanics. The tracheo-bronchial tree and the arterial tract are characterized by numerous sites of curvature and branching. The flow is thus three-dimensional and never fully developed. Simple models are necessary to determine the main features of the flow in such a complex geometry. Numerical simulations of physiological flows, once validated, are developed to assess the temporal and spatial variations of the parameters of interest, mainly the velocity field. Furthermore, the effect of governing factors is easily studied, other quantities remaining constant. However, any theoretical model deals at best with pressure boundary conditions rather than velocity conditions only, owing to the developing nature of the flow.

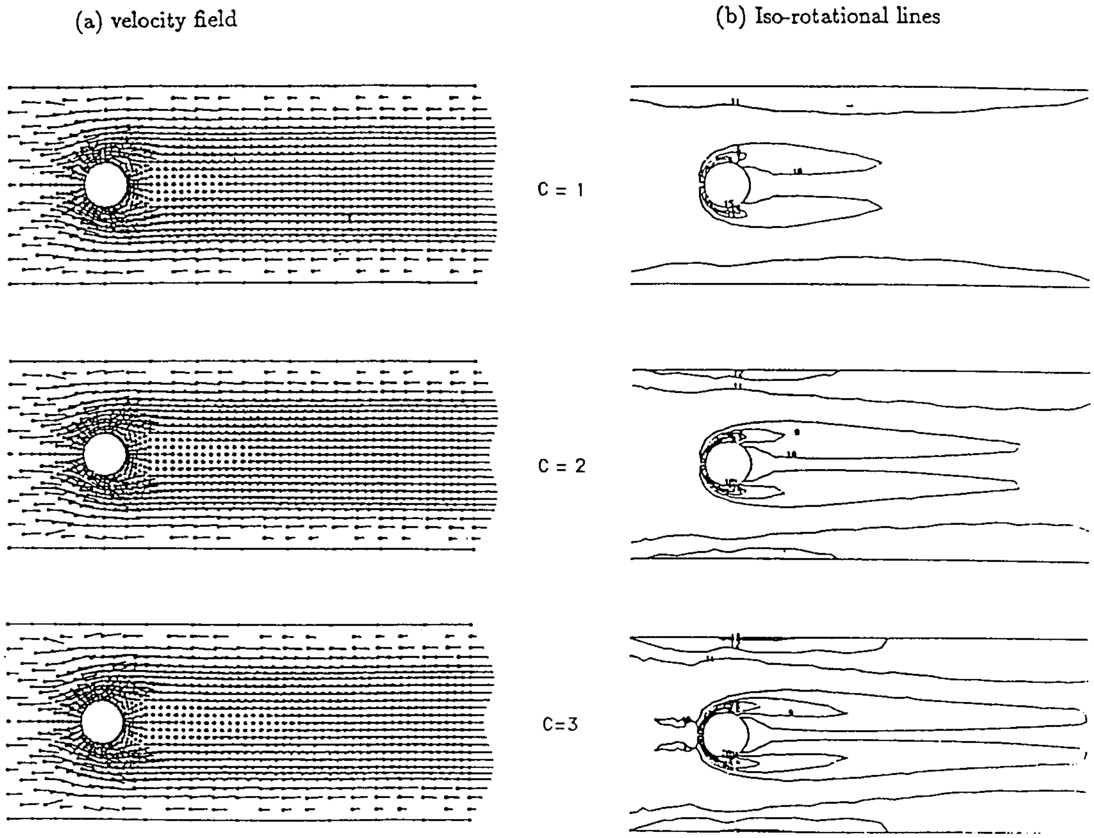

Figure 3. Flow around a cylinder in a channel (computed by C. Bègue). A pressure difference C is imposed at both ends of the channel. Thus the velocity at the outlet is a result of the computation. Re = 40

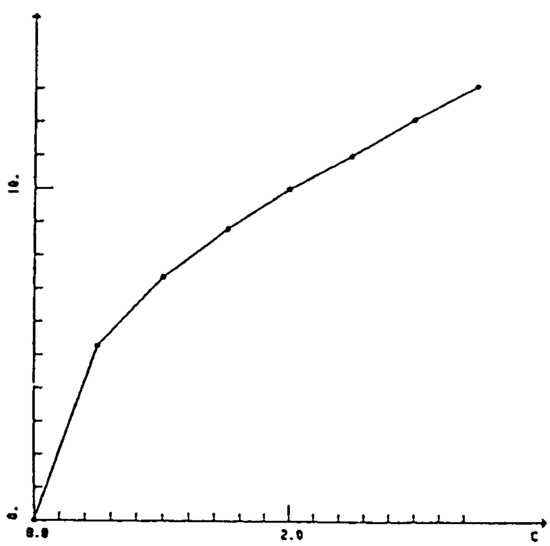

Figure 4. Variation in the flux across the channel corresponding to Figure 3 when C varies (computed by C. Bègue)

Steady flow is always the first reference step of any study. However, physiological flows are unsteady with respect to both time and wall motion. Pressure boundary conditions are thus essential.

A uniform pressure setting at the inlet and outlet of the pipe network avoids any *a priori* knowledge of the fluid velocity.

Two examples of 3D flow illustrate numerical simulations performed with imposed pressure at the outlet and/or inlet sections of the fluid domain. The first example deals with steady flow in a model of the aortic bifurcation, which includes a tapered transition zone. The second example concerns oscillatory flows in a 90° bend. In both examples the pressure was set to zero at the outlet stations located far downstream from the test section, taking into account the inertia effect.

In the unsteady flow experiments, the inlet boundary condition was a time-varying pressure associated with a null cross-flow.

4.2.2. Discretization. The equations to solve are

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \times [0,\ T],$$

$$p = p_i \quad \text{on}\ \Gamma_i, \quad i = 1,\ 2, \ldots,\ N,$$

$$\mathbf{u} \times n = 0 \quad \text{on}\ \Gamma_i, \quad i = 1,\ 2, \ldots,\ N,$$

$$\mathbf{u} = 0 \quad \text{on}\ \Gamma_0,$$

where Γ_1, $\Gamma\ 2$, ..., Γ_N are the inlet or outlet boundaries and Γ_0 represents the walls.

Remark. A static pressure is imposed instead of a dynamic one. This is not supported by the theory, but this is a limitation of the theory rather than a mistake. Here at each time step we have a Stokes problem for which the theory applies.

To discretize in time, we use the method of characteristics. Recall that this method consists of approaching the non-linear term of the equation by

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \approx \frac{\mathbf{u}^{n+1}(x) - \mathbf{u}^n(X^n(x))}{\delta t},$$

where δt is the time step, $\mathbf{u}^n$ is the velocity field at time $t^n = n\delta t$ and $X^n(x)$ is the position at time t^n of the particle which is at x at time t^{n+1}.

The function X^n is approached by integrating the ordinary differential system

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \mathbf{u}_h^n(X(t),\ t), \qquad X(t^{n+1}) = x$$

using Euler's method ($\mathbf{u}_h^n$ is the approximation of $\mathbf{u}^n$ obtained at time level $n\delta t$) (see e.g. Reference 16).

At each time iteration one must solve a PDE system of the generalized Stokes type

$$\frac{1}{\delta t}\ \mathbf{u}^n - \nu \Delta \mathbf{u} + \nabla p = f \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{on}\ \Omega,$$

$$p = p_i \quad \text{on}\ \Gamma_i, \quad i = 1,\ 2, \ldots,\ N,$$
$$\mathbf{u} \times n = 0 \quad \text{on}\ \Gamma_i, \quad i = 1,\ 2, \ldots,\ N,$$
$$\mathbf{u} = 0 \quad \text{on}\ \Gamma_0,$$

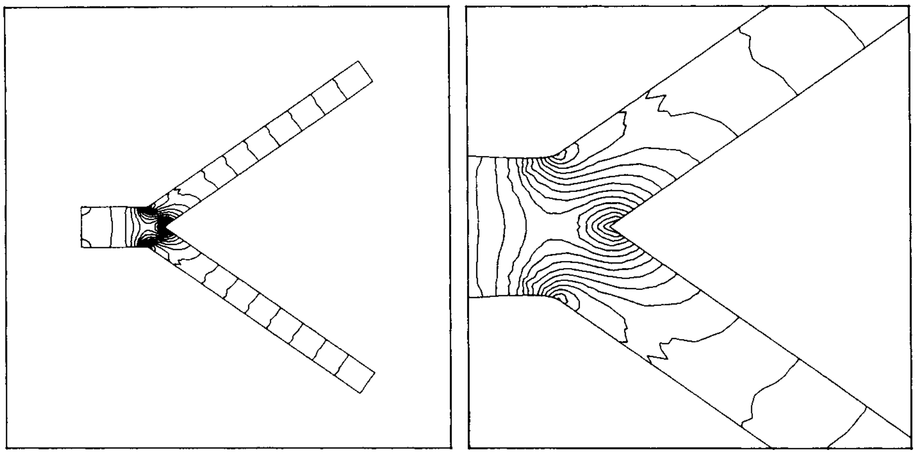

Figure 5. Pressure map in a symmetric bifurcation (area ratio of 0·8, bifurcation angle of 70°) at $Re = 1200$: left, in the whole centreplane; right, zoom on the transition zone and the entrance region of the branches. Equal pressures are imposed on the outlet boundaries

which is approached by the numerical scheme

$$a_h(u_h, v_h) + b(v_h, p_h) = (f, v_h) \quad \forall v_h \in V_h,$$

$$b(u_h, q_h) = 0 \quad \forall q_h \in Q_h,$$

where

$$a_h(u_h, v_h) = \nu \int_\Omega \nabla_h \cdot \nabla_h \, dx + \nu \int_\Omega \nabla \cdot u_h \nabla \cdot v_h \, dx,$$

$$b(v_h, p_h) = - \int_\Omega \nabla \cdot v_h p_h \, dx,$$

$$V_h = \{v_h \in X_h : v_h \times n = 0 \quad \text{on } \Gamma_i, \ i = 1, \ldots, N; \ v_h = 0 \quad \text{on } \Gamma_0\},$$

$$Q_{h0} = \{q_h \in M_h : q_h = 0 \quad \text{on } \Gamma_i, \ i = 1, \ldots, N\}.$$

Here X_h is the space of P^1 + bubble finite elements and M_h is the space of P^1 continuous functions, i.e. the degree of freedom of X_h are the values of v_h at the vertex and the barycentre of each tetrahedron of the mesh and the degrees of freedom of Q_h are the values at the vertex (see e.g. Reference 16).

Figures 5 and 6 show the pressure variations in the centreplane of a bifurcation. The bifurcation pipe is composed of circular pipes, a parent tube (on the left of the figures) and two branches (on the right), and a transition zone in the middle computed by a smoothing spline (bifurcation angle of 70°, area ratio of 0·8). The Reynolds number based on the parent tube diameter and the mean velocity is equal to 1200. In both cases a constant velocity profile was imposed at the inlet and atmospheric pressure on the outlet of the upper branch. In Figure 6 the pressure on the upper branch is equal to that on the lower branch. Figure 6 shows the effect of a pressure change in one branch of the bifurcation. The flow is blocked in the lower branch (the pressure is much larger. Flow separation occurs only in the first case (Figure 5) in the entrance region of both branches.

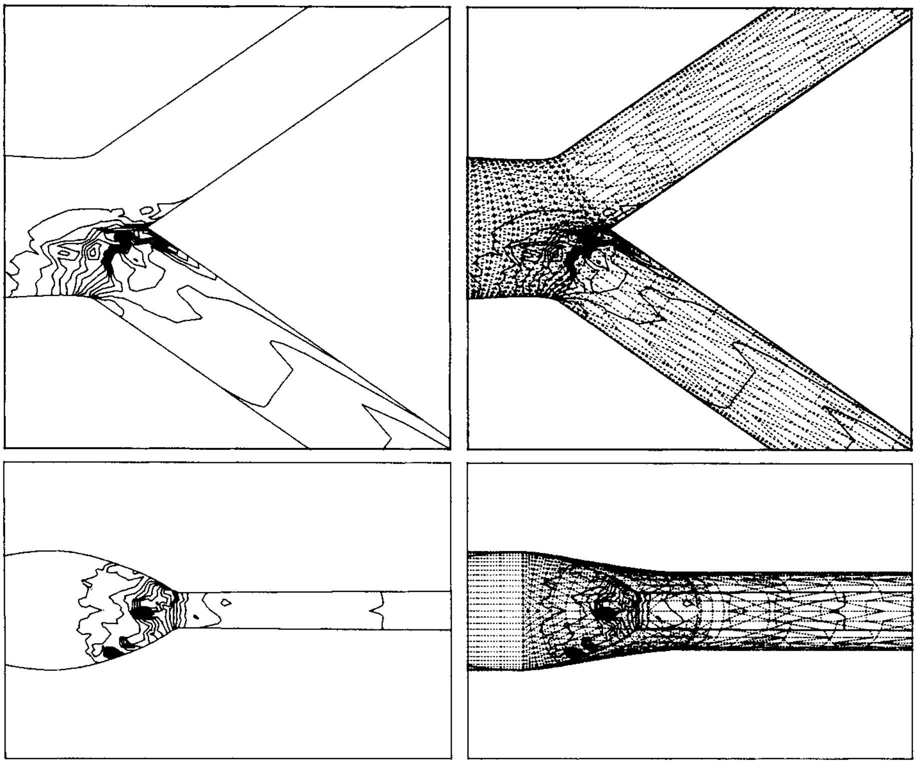

Figure 6. Same as Figure 5 but with a very high pressure imposed on the lower outlet, which blocks the flow in the corresponding branch. The pressure map is displayed in the centreplane (upper panels), with the wall mesh (right), and in a plane normal to the centreplane, parallel to the axis of the lower tube and near the lower part of the wall (bottom panels)

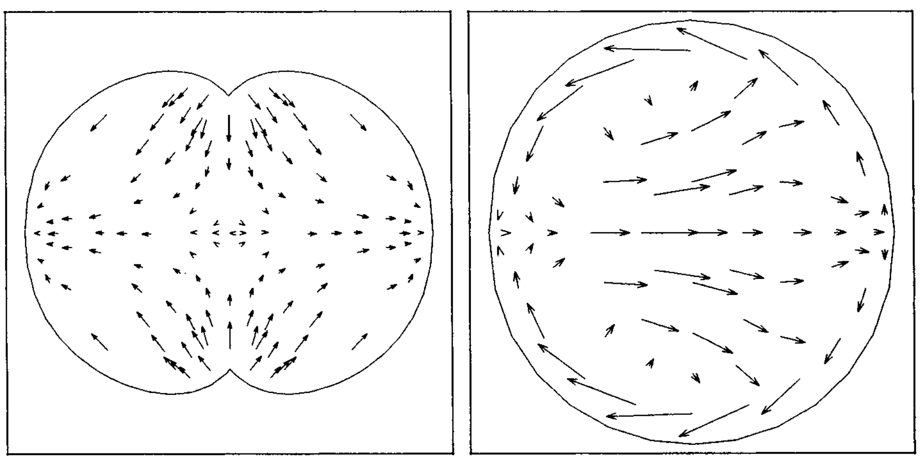

Figure 7. Secondary flows in the bifurcation pipe of Figure 5. Cross-velocity vectors are shown in the cross-section located 0·75d downstream from the inlet of the corresponding segment: left, transition zone; right, entrance region of the branch

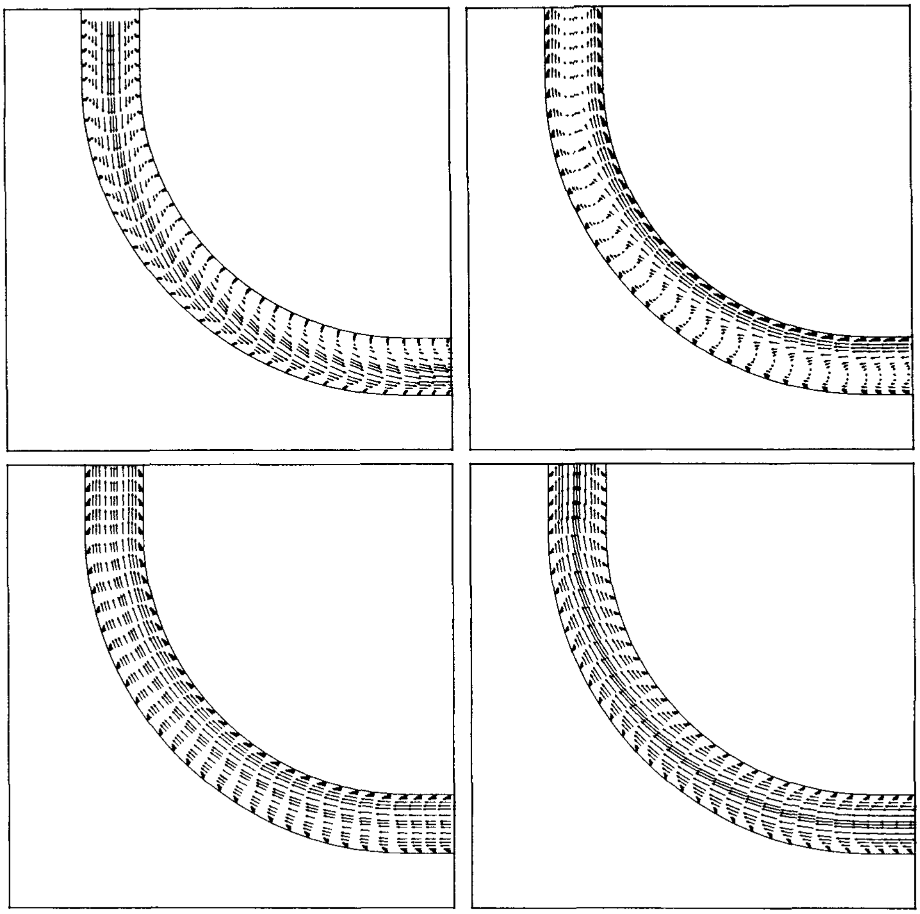

Figure 8. Axial velocity vectors in the centreplane of a time-dependent flow in a torus-shaped pipe. A sinusoidal wave pressure (frequency of 1 Hz) is prescribed at the inlet. Each view is separated by 0·1 s and the views are in the sequence left to right and top to bottom. The average Reynolds number equal 500 and the Stokes number $R_{tube}(\omega/\nu)^{1/2} = 5$. Each vector shown corresponds to one vertex of the triangulation in the central plane

The velocity vector of a three-dimensional flow can be decomposed into two components, one in the axial direction of the pipe and the other at right angles to the tube axis, the so-called secondary flow. Secondary flows are shown in Figure 7. Two kinds of secondary flows are observed, namely the source–sink cross-flow in the transition zone and the bend transverse motion in the branch.[17]

The second set of numerical experiments is performed on a 90° bend; the curvature ratio is 1:10. The Reynolds number based on the tube diameter and the mean velocity is equal to 500. The pressure is imposed on both ends of the tube. The pressure difference is a sinusoidal wave in time, with frequency equal to 1 Hz. Figure 8 shows axial velocity vectors in the curvature plane at four instants of time separated by 0·1 s before and after the change in direction of the flow in the tube centre.

Before the complete change in direction the region of back flow near the inner wall is much wider than in a straight pipe of similar size, whereas near the outer edge it is narrower. Another interesting feature is the motion of the peak axial velocity from the inner to the outer wall. Such a property has been confirmed by experiments using nuclear magnetic resonance velocimetry.

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